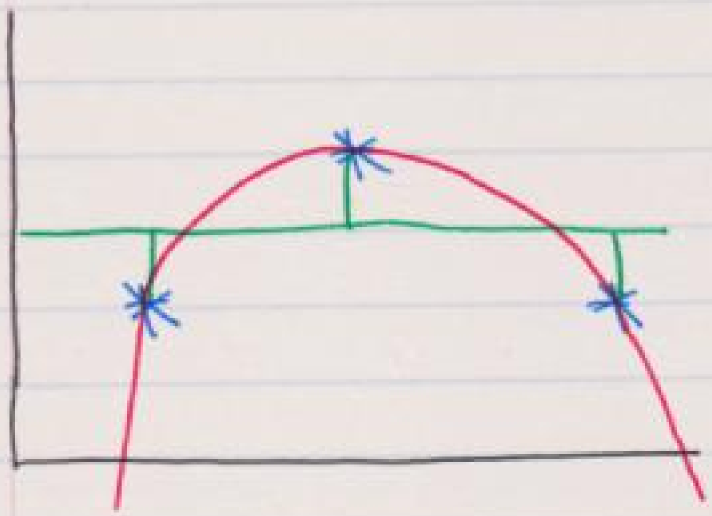
CSCC37 Approximation/ Interpolation Notes

Introduction:

- With approximation, the line does not go through all the points on the graph.

- With interpolation, the line goes through the points on the graph.

E.g.



The red line is interpolation.

The green line is approximation.

- Truncated Taylor Series:

$$p(x) = \underbrace{F(a) + F'(a)(x-a) + \dots + \frac{F^{(n)}(a)}{n!}(x-a)^n}_{\text{Only has the first } n+1 \text{ terms}}$$

This is polynomial because of the $(x-a)^i$, $i=0,1,\dots$

The error $e(x) = p(x) - F(x)$

$$= \frac{F^{(n+1)}(\eta)}{(n+1)!}(x-a)^{n+1}$$

- Some other approximations are:

a) Interpolation: Find a polynomial p s.t. $p(x_i) = F(x_i)$, $i = 0, 1, 2, \ldots$
F is the function we're trying to approximate. It could simply be a set of data.

b) Least Squares: Find a polynomial p s.t. $p(x)$ minimize $\|F-p\|_2 = \left(\int_a^b (F(x)-p(x))^2\,dx\right)^{1/2}$

Other norms we can use for least squares are:

i) $\|F-p\|_\infty = \max_{a \le x \le b} |F(x)-p(x)|$

ii) $\|F-p\|_1 = \int_a^b |F(x)-p(x)|\,dx$

Note: If you want to approximate a function around a given point and you have access to derivatives of the function, then you may want to use a Taylor expansion. If you want to approximate a function on an interval where you can access some function values but not derivatives, you can use an interpolation polynomial.

Polynomial Interpolation:

- Consider P_n, which is the set of polynomials of degree $\le n$. This is a function space and requires the basis of $n+1$ functions. The most common basis is the monomial basis, which is $\{x^i, i = 0, 1, 2, \ldots\}$.

Weierstrass' Thm:

- If a function F is continuous on an interval $[a,b]$, then for any $\varepsilon > 0$, $\exists\, p_\varepsilon$ s.t. $\|F - p_\varepsilon\| < \varepsilon$.
- This means that for any continuous function on a closed interval $[a,b]$, there exists some polynomial that is as close to it as it can be.

Numerical Methods For Polynomial Interpolation:

1. Vandermonde Thm:

- Also known as Method of Undetermined Coefficients.
- Thm: For any sets $\{x_i, i = 0, 1, \dots n\}$ and $\{y_i, i = 0, 1, \dots, n\}$, for distinct x_i's and undistinct y_i's, $\exists$ a unique polynomial $P(x) \in P_n$ s.t. $P(x_i) = y_i$, $i = 0, 1, \dots n$.
- Proof:

If $P(x)$ exists, then it must be possible to write it as

$$P(x) = \sum_{i=0}^{n} a_i x^i$$

This can be converted into a matrix problem with $P(x_i) = y_i$, $i = 0, 1, 2, \dots n$.

We can solve for the a_i's using the Vandermonde Matrix.

$$\underbrace{\begin{bmatrix} (x_0)^0 & (x_0)^1 & (x_0)^2 & \cdots & (x_0)^n \\ (x_1)^0 & (x_1)^1 & (x_1)^2 & \cdots & (x_1)^n \\ \vdots & \vdots & \vdots & & \vdots \\ (x_n)^0 & (x_n)^1 & (x_n)^2 & \cdots & (x_n)^n \end{bmatrix}}_{\text{Vandermonde Matrix}} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$$

The question now becomes: "Is the Vandermonde Matrix non-singular?"

The Vandermonde matrix is non-singular because all the columns are linearly independent.

- The Vandermonde Theorem proves existence but does not lead to the best algorithm. It can be poorly conditioned.
- Gives the monomial basis.

2. Lagrange Basis:

- For a simple interpolation problem $P(x_i) = y_i$, $i = 0, 1, \dots n$, consider the basis

$$l_i = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} \quad \text{for } i = 0, 1, 2, \dots, n$$

$$= \left(\frac{x - x_0}{x_i - x_0}\right) \cdots \underbrace{\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)\left(\frac{x - x_{i+1}}{x_i - x_{i+1}}\right)}_{} \cdots \left(\frac{x - x_n}{x_i - x_n}\right)$$

Notice that we skipped $\frac{x - x_i}{x_i - x_i}$.

- $l_i(x) \in P_n$
- Consider $l_i(x_j)$.

If $j=i$, we get $\prod_{\substack{j=i \\ j\neq i}}^{n} \frac{x_i - x_j}{x_i - x_j} = 1$ $\quad j=i$, $x_j = x_i$

I.e. $l_i(x_j), j=i, = \prod_{\substack{j=i \\ j\neq i}}^{n} \frac{x_i - x_j}{x_i - x_j}$

$$= \left(\frac{x_i - x_0}{x_i - x_0}\right) \cdots \left(\frac{x_i - x_{i-1}}{x_i - x_{i-1}}\right) \left(\frac{x_i - x_{i+1}}{x_i - x_{i+1}}\right) \cdots \left(\frac{x_i - x_n}{x_i - x_n}\right)$$

$$= 1$$

If $j \neq i$, we get $l_i(x_j), j\neq i = 0$.

$$\prod_{\substack{j\neq i \\ j\neq i}}^{n} \frac{x_j - x_j}{x_i - x_j} = 0$$

Expanding the product above, we get

$$\left(\frac{x_j - x_0}{x_i - x_0}\right) \cdots \left(\frac{x_j - x_{i-1}}{x_i - x_{i-1}}\right)\left(\frac{x_j - x_{i+1}}{x_i - x_{i+1}}\right) \cdots \left(\frac{x_j - x_n}{x_i - x_n}\right)$$

One of these products will be 0 as $0 \leq j \leq n$, and $j \neq i$. Hence, the entire product will be 0.

To summarize, $l_i(x_j) = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{if } i \neq j \end{cases}$

- The lagrange polynomial is free to construct, but very expensive to evaluate at non-interpolation points.

- With the basis function, we can write out the interpolating polynomial for free.

$$P(x) = \sum_{i=0}^{n} l_i(x) y_i$$

Furthermore, $P(x_i) = y_i$, for $i=0,1,\ldots,n$ because

$$P(x_i) = \sum_{i=0}^{n} \underbrace{l_i(x_i)}_{\Downarrow} y_i$$

Equals to 1, as stated previously

3. Newton Basis:

- Also called Divided Differences.
- For a simple interpolation $P(x_i) = y_i$, $i=0,1,\ldots n$, we look for an interpolating of the form

$$P(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

Converting into a matrix, we get

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & x_1-x_0 & 0 & \cdots & 0 \\ 1 & x_2-x_0 & (x_2-x_0)(x_2-x_1) & & 0 \\ \vdots & & & & \vdots \\ 1 & x_n-x_0 & \cdots & & \prod_{i=0}^{n-1}(x_n-x_i) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$$

This is a lower triangular matrix, meaning that no factorization is involved.

$$\left.\begin{aligned} a_0 &= y_0 \\ a_1 &= \frac{y_1-y_0}{x_1-x_0} \\ a_2 &= \frac{\frac{y_2-y_1}{x_2-x_1} - \frac{y_1-y_0}{x_1-x_0}}{x_2-x_0} \end{aligned}\right\} \text{Divided Differences}$$

- Divided Differences: $Y[x_i] = y(x_i) = y_i$

$$Y[x_{i+k}, \ldots, x_i] = \frac{Y[x_{i+k}, \ldots, x_{i+1}] - Y[x_{i+k-1}, \ldots, x_i]}{x_{i+k} - x_i}$$

E.g. $Y[x_2, x_1, x_0] = \dfrac{Y[x_2, x_1] - Y[x_1, x_0]}{x_2 - x_0}$

– Newton's Polynomial: $p(x) = Y[x_0] + (x-x_0)Y[x_1, x_0] + \dots + (x-x_0)(x-x_1)\dots(x-x_{n-1})y[x_n, \dots, x_0]$

Then, $p(x) \in P_n$ and $p(x_i) = y_i$, $i = 0, 1, 2, \dots, n$

E.g. Find a $P \in P_3$ s.t. $P(0)=1$, $P(1)=3$, $P(2)=9$, $P(3)=25$

Soln:

X	$Y[x_i]$	$Y[x_{i+1}, x_i]$	$Y[x_{i+2}, \dots, x_i]$	$Y[x_{i+3}, \dots, x_i]$
0	1			
		$\frac{3-1}{1-0} = 2$		
			$\frac{6-2}{2-0} = 2$	
1	3			$\frac{5-2}{3-0} = 1$
		$\frac{9-3}{2-1} = 6$		
			$\frac{16-6}{2-1} = 5$	
2	9			
		$\frac{25-9}{3-2} = 16$		
3	25			

$$P(x) = y[x_0] + (x-x_0)y[x_1, x_0] + (x-x_0)(x-x_1)\,y[x_2, x_1, x_0] + (x-x_0)(x-x_1)(x-x_2)\,y[x_3, \dots, x_0]$$

$$= 1 + 2x + 2x(x-1) + x(x-1)(x-2)$$

Read coefficients from top of triangle.

– How are divided differences and derivatives related?

Consider $y[x_1, x_0] = \dfrac{y(x_1) - y(x_0)}{x_1 - x_0}$

$$\lim_{x_1 \to x_0} y[x_1, x_0] = \lim_{x_1 \to x_0} \frac{y(x_1) - y(x_0)}{x_1 - x_0}$$

$= y'(0)$, provided that $y'(x_0)$ exists

Consider $y[x_2, x_1, x_0] = \dfrac{y[x_2, x_1] - y[x_1, x_0]}{x_2 - x_0}$

$$\lim_{\substack{x_2 \to x_0 \\ x_1 \to x_0}} y[x_2, x_1, x_0] = \frac{y''(x_0)}{2!}$$

In general, we can show that

$$\lim_{\substack{x_k \to x_0 \\ x_{k-1} \to x_0 \\ \vdots \\ x_1 \to x_0}} y[x_k, \dots, x_0] = \frac{y^{(k)}(x_0)}{k!}$$

– How does this help with osculatory interpolation, which is interpolation with derivatives?

E.g. Find $P \in P_4$ s.t. $P(0)=0$, $P(1)=1$, $P'(1)=1$, $P''(1)=2$ and $P(2)=6$.

Soln:

x_i	$Y[x_i]$	$Y[x_{i+1}, x_i]$	$Y[x_{i+2}, \dots, x_i]$	$Y[x_{i+3}, \dots, x_i]$	$Y[x_{i+4}, \dots, x_i]$
0	0				
		$\frac{1-0}{1-0}=1$			
			$\frac{1-1}{1-0}=0$		
1	1			$\frac{1-0}{1}=1$	
		$\frac{y'(1)}{1!}=1$			$\frac{3-1}{2-0}=1$
1	1		$\frac{y''(1)}{2!}=1$		
		$\frac{y'(1)}{1!}=1$		$\frac{4-1}{1}=3$	
1	1		$\frac{5-1}{2-1}=4$		
		$\frac{6-1}{2-1}=5$			
2	6				

$$P(x) = Y[0] + xY[1,0] + x(x-1)Y[1,1,0] + x(x-1)^2 Y[1,1,1,0] + x(x-1)^3 Y[2,1,1,1,0]$$
$$= 0 + x + x(x-1)^2 + x(x-1)^3$$ ← Read the coefficients from top of triangle.

Error in Polynomial Interpolation:

- $E(x) = \underbrace{Y(x)}_{\text{Underlying Function}} - \underbrace{P(x)}_{\text{Interpolating Polynomial}}$

- For a simple interpolation $P(x_i) = Y_i$, $i = 0, 1, 2, \dots, n$ we can show that $E(x) = \frac{y^{(n+1)}}{(n+1)!}(\xi) \prod_{i=0}^{n} (x - x_i)$

where $\xi \in \text{span}\{x_0, \dots, x_n, x\}$

$= [\min\{x_0, \dots, x_n, x\}, \max\{x_0, \dots, x_n, x\}]$